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Matchings in regular graphs from eigenvalues

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ABSTRACT

Let G be a connected k-regular graph of order n. We find a best upper bound (in terms of k) on the third largest eigenvalue that is sufficient to guarantee that G has a perfect matching when n is even, and a matching of order $n-1$ when n is odd. We also examine how other eigenvalues affect the size of matchings in G.

1. Introduction

Throughout, G denotes a simple graph of *order* n (the number of vertices) and *size* e (the number of edges). The *eigenvalues* of G are the eigenvalues λ_i of its adjacency matrix A, indexed so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The greatest eigenvalue, λ_1, is also called the *spectral radius*. If G is k-regular, then it is easy to see that $\lambda_1 = k$ and that $\lambda_2 < k$ if and only if G is connected.

The eigenvalues of a graph are related to many of its properties and key parameters. The most studied eigenvalues have been the spectral radius λ_1 (in connection with the chromatic number, the independence number and the clique number of the graph [13,14,19,22]), λ_2 (in connection with the expansion property of the graph [15]) and λ_n (in connection with the chromatic and the independence number of the graph [14] and the maximum cut [17]). We refer the reader to the monographs [5,9,10,12] as well as the recent surveys [15,17] for more details about eigenvalues of graphs and their applications.

In this paper, we relate the eigenvalues of a connected regular graph G to its *matching number*, $\nu(G)$, the maximum size of a matching in G. This relationship was initiated in [3] by Brouwer and

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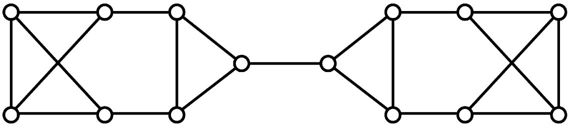

Fig. 1. A 3-regular graph with $\lambda_2 > \theta > \lambda_3$.

Haemers who gave sufficient conditions for the existence of a perfect matching in a graph in terms of its Laplacian eigenvalues and, for a regular graph, gave an improvement in terms of the third largest adjacency eigenvalue, λ_3. Their result in [3] on perfect matchings was improved in [6] and extended in [8] to obtain lower bounds on $\nu(G)$. The results presented here further improve those in [8]. The improvements are stated in terms of an explicitly determined function $\rho(k)$. In many cases, $\rho(k)$ is proved to be the best possible upper bound that is a function of k only.

The function $\rho(k)$ is initially defined as follows. Let $\mathcal{H}(k)$ denote the class of all connected irregular graphs with maximum degree k, odd order n, and size e with $2e \geqslant kn - k + 2$. Suppose also that each graph in $\mathcal{H}(k)$ has at least 4 vertices of maximum degree k if k is odd and at least 3 if k is even. We define $\rho(k)$ to be the minimum of the spectral radii of the graphs H in $\mathcal{H}(k)$:

$$\rho(k) := \min_{H \in \mathcal{H}(k)} \lambda_1(H). \tag{1}$$

We are now able to state our main theorem, proved in Section 2. We assume throughout that $k \geqslant 3$, because a connected k-regular graph G with $k = 1$ or 2 is either a single edge or a cycle and $\nu(G)$ is easily determined.

Theorem 1. *Let G be a connected k-regular graph of order n such that*

$$\lambda_3(G) < \rho(k).$$

Then $\nu(G) = \lfloor n/2 \rfloor$. That is,

(1) *if n is even, G contains a perfect matching*;
(2) *if n is odd, G contains a matching on $n - 1$ of its vertices.*

It easily follows that $\rho(k) > k - 1$, hence $\lambda_3(G) \leqslant k - 1$ is sufficient in Theorem 1. To get a best possible bound, explicit expressions for $\rho(k)$ are needed. These will be obtained in the following theorem, proved in Section 3.

Theorem 2. *Let θ denote the greatest solution of $x^3 - x^2 - 6x + 2 = 0$. Then*

$$\rho(k) = \begin{cases} \theta = 2.85577\ldots & \text{if } k = 3, \\ \frac{1}{2}\left(k - 2 + \sqrt{k^2 + 12}\right) & \text{if } k \geqslant 4 \text{ is even}, \\ \frac{1}{2}\left(k - 3 + \sqrt{(k+1)^2 + 16}\right) & \text{if } k \geqslant 5 \text{ is odd}. \end{cases}$$

In Lemma 8 in Section 5, we show that the upper bound in Theorem 1 is the best possible function of k by presenting for each $k \geqslant 3$, examples of k-regular graphs $G(k)$ of even order with no perfect matching and with $\lambda_3(G(k)) = \rho(k)$.

Note that Theorem 1 is also true if the condition $\lambda_3(G) < \rho(k)$ is replaced by the more restrictive condition $\lambda_2(G) < \rho(k)$. The condition $\lambda_2(G) < \rho(k)$ is perhaps a more natural one since it involves the more commonly studied *spectral gap* $k - \lambda_2$. However, there are perfect matchings that are detected by the condition $\lambda_3(G) < \rho(k)$, but missed by the more demanding condition $\lambda_2(G) < \rho(k)$. For example, for the case when n is even, the 3-regular graph in Fig. 1 has $\lambda_3 < 2.12 < \theta$ and so has a perfect matching, but $\lambda_2 > 2.87 > \theta$.

A comparison of the following theorem with Theorem 1 shows that when n is odd, the condition $\lambda_2(G) < \rho(k)$ is considerably more restrictive than the condition $\lambda_3(G) < \rho(k)$. The theorem is proved in Section 4. In Lemma 9 in Section 5, the bound $\rho(k)$ is shown to be the best possible function of k.

Theorem 3. *If G is a connected k-regular graph of odd order n such that*

$$\lambda_2(G) < \rho(k),$$

then for each vertex x, $G\backslash\{x\}$ contains a perfect matching.

We conclude the introduction by noting that Theorem 1 implies a corollary on the number of edge-disjoint matchings in a regular graph of even order. The corollary was first stated in [3, Corollary 3.3] in terms of Laplacian eigenvalues.

Corollary 4. *A k-regular graph G of even order has at least $\lfloor \frac{k-\lambda_2(G)+1}{2} \rfloor$ edge-disjoint perfect matchings.*

Proof. If G is a k-regular graph of even order with $\lambda_2(G) \leqslant k-1$, then G is connected. Also, $k-1 < \rho(k)$ so Theorem 1 implies that G has a perfect matching M. Deleting M from G yields a $(k-1)$-regular graph $G-M$ with $\lambda_2(G-M) \leqslant \lambda_2(G)+1$ [16, p. 181]. Also, if $\lambda_2(G) \leqslant k-3$ then $\lambda_2(G-M) \leqslant k-2$ so $G-M$ will be connected and will also have a perfect matching by Theorem 1. Repeating this observation, we see that if t is a positive integer such that $\lambda_2(G) \leqslant k-2t+1$, then G has t edge-disjoint perfect matchings. $\square$

2. The proof of Theorem 1

The proof of Theorem 1 will follow immediately from Lemma 5 below. The lemma was first proved for graphs of even order by Brouwer and Haemers in [3, Theorem 3.1]. In Lemma 5, n may be odd.

Lemma 5. *Let G be a connected k-regular graph on n vertices where $k \geqslant 3$. If*

$$\nu(G) \leqslant \frac{n-2}{2},$$

then G has 3 vertex disjoint induced subgraphs H_1, H_2, H_3 in $\mathcal{H}(k)$.

Proof. As in [8], we use the Berge–Tutte formula which asserts (see [1] or [21, p. 139]) that

$$\nu(G) = \frac{1}{2}\Big(n + \min_{S\subset V(G)}\big(|S| - \operatorname{odd}(G\setminus S)\big)\Big)$$

where $\operatorname{odd}(G\setminus S)$ denotes the number of odd components of $G\setminus S$.

Suppose that $\nu(G) \leqslant \frac{n-2}{2}$. By the Berge–Tutte formula, it follows that there is a subset S such that $2\nu = n+s-q$ where $s=|S|$ and $q=\operatorname{odd}(G\setminus S)$. Thus $q \geqslant s+2$. Note that $s>0$, otherwise $\nu(G) = \frac{1}{2}(n-\operatorname{odd}(G)) = \frac{n-1}{2} > \frac{n-2}{2}$, a contradiction. Let $H_1,\dots,H_q$ denote the odd components of $G\setminus S$. Denote by n_i and e_i the order and the size of H_i respectively.

For $i\in\{1,2,\dots,q\}$, denote by t_i the number of edges with one endpoint in H_i and the other in S. Because G is connected, it follows that $t_i \geqslant 1$ for each $i\in\{1,2,\dots,q\}$. Also, since vertices in H_i are adjacent only to vertices in H_i or S, we deduce that $2e_i = kn_i - t_i = k(n_i-1)+k-t_i$. Because n_i is odd, it follows that $k-t_i$ is even. Thus, t_i has the same parity as k for each $i\in\{1,2,\dots,q\}$.

The sum of the degrees of the vertices in S is at least the number of edges between S and $\bigcup_{i=1}^q H_i$. Thus, $ks \geqslant \sum_{i=1}^q t_i$. Since $q\geqslant s+2$, there are at least 3 t_i's such that $t_i<k$; otherwise, there are $q-2$ t_i's such that $t_i\geqslant k$ and so $ks \geqslant \sum_{i=1}^q t_i \geqslant k(q-2)+2 \geqslant ks+2$, a contradiction. Because t_i and k have the same parity, this implies there are at least 3 t_i's, t_1,t_2,t_3 say, satisfying $t_i\leqslant k-2$.

Suppose now that $i\in\{1,2,3\}$. Then, $t_i\leqslant k-2$, so $n_i>1$ and $2e_i = kn_i - t_i \geqslant kn_i-k+2$. Also, $n_i(n_i-1)\geqslant 2e_i \geqslant kn_i-k+2$, so $n_i \geqslant k+2/(n_i-1)$. Hence, $n_i\geqslant k+2\geqslant t_i+4$ if k is odd and

$n_i \geqslant k+1 \geqslant t_i+3$ if k is even. Thus each of the odd components H_i, $i \in \{1,2,3\}$, has at least 4 vertices of degree k if k is odd, and at least 3 if k is even. Thus, $H_1, H_2, H_3 \in \mathcal{H}(k)$, as required. □

Proof of Theorem 1. Suppose now that G satisfies the conditions in Theorem 1 and $\lambda_3(G) < \rho(k)$. If $\nu(G) \leqslant \frac{n-2}{2}$ then, by Lemma 5, G has 3 vertex disjoint induced subgraphs H_1, H_2, H_3 in $\mathcal{H}(k)$. Consequently, by the inclusion principle [16, p. 189],

$$\lambda_3(G) \geqslant \lambda_3(H_1 \cup H_2 \cup H_3) \geqslant \min_i \lambda_1(H_i) \geqslant \rho(k), \tag{2}$$

a contradiction. Thus $\nu(G) > \frac{n-2}{2}$ and the statements in Theorem 1 follow. □

3. The proof of Theorem 2: The formula for $\rho(k)$

It remains to determine the explicit formulas for the function $\rho(k)$ given in Theorem 2. Recall that $\rho(k)$ is defined in (1) as the minimum of the spectral radii of the graphs $H \in \mathcal{H}(k)$. We begin by proposing a candidate graph $H(k)$ that minimizes the spectral radius for each of the three cases in Theorem 2.

Let C_n and K_n denote a cycle and a complete graph of order n, respectively, and, for n even, let M_n denote a matching on n vertices. Also, let $\overline{G}$ denote the complement of a graph G. If G_1 and G_2 are vertex disjoint graphs, let their *join* $G_1 \vee G_2$ be the graph G formed from G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2.

Let G_5 denote the graph of order 5 obtained from K_4 by subdividing one of its edges by a new vertex. It is straightforward to check that if

$$H(k) = \begin{cases} G_5 & \text{if } k = 3, \\ K_3 \vee \overline{M}_{k-2} & \text{if } k \geqslant 4 \text{ is even}, \\ C_4 \vee \overline{C}_{k-2} & \text{if } k \geqslant 5 \text{ is odd}. \end{cases} \tag{3}$$

then $H(k) \in \mathcal{H}(k)$ and so is a candidate for a graph in $\mathcal{H}(k)$ of minimum spectral radius.

In this section, to prove Theorem 2, we first show (in Lemma 6) that the spectral radius of $H(k)$ is given by the formulas in Theorem 2 and then prove in Lemma 7, that to show that $H(k)$ has minimum spectral radius, we need only compare it with graphs $H \in \mathcal{H}(k)$ of a specific order and size. The proof of Theorem 2 will then be reduced to showing that $\rho(H(k)) \leqslant \lambda_1(H)$ for all such graphs H. Our arguments will require frequent use of inequality (4) described below.

Suppose that $V = V_1 \cup V_2$ is a partition of the vertex set V of a graph G of order n and size e. For $i = 1, 2$, let G_i be the subgraph of G induced by V_i, and let n_i and e_i be the order and size, respectively, of G_i. Also, let G_{12} be the bipartite subgraph induced by the partition and let e_{12} be the size of G_{12}. A theorem of Haemers [13] shows that the eigenvalues of the quotient matrix of the partition interlace the eigenvalues of the adjacency matrix of G (see also Godsil and Royle [12, p. 197]). Here the quotient matrix is

$$Q = \begin{bmatrix} \frac{2e_1}{n_1} & \frac{e_{12}}{n_1} \\ \frac{e_{12}}{n_2} & \frac{2e_2}{n_2} \end{bmatrix}.$$

Applying this result to the greatest eigenvalue of G, we get

$$\lambda_1(G) \geqslant \lambda_1(Q) = \frac{e_1}{n_1} + \frac{e_2}{n_2} + \sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{e_{12}^2}{n_1 n_2}} \tag{4}$$

with equality if and only if the partition is *equitable* [12, p. 195]; equivalently, if and only if G_1 and G_2 are regular, and G_{12} is semiregular.

Lemma 6. *Let θ denote the greatest solution of $x^3 - x^2 - 6x + 2 = 0$ and let $H(k)$ be defined as in (3). Then*

$$\lambda_1\big(H(k)\big) = \begin{cases} \theta = 2.85577\ldots & \text{if } k = 3, \\ \frac{1}{2}(k - 2 + \sqrt{k^2 + 12}) < k - 1 + \frac{3}{k} & \text{if } k \geqslant 4 \text{ is even}, \\ \frac{1}{2}(k - 3 + \sqrt{(k+1)^2 + 16}) < k - 1 + \frac{4}{k+1} & \text{if } k \geqslant 5 \text{ is odd}. \end{cases}$$

Proof. The case $k = 3$ follows by showing, for example, that $x^3 - x^2 - 6x + 2$ is the characteristic polynomial of the quotient matrix of an equitable three part partition of $H(3)$. The two remaining expressions for $\lambda_1(H(k))$ are obtained by applying formula (4) to the graphs $H(k) = K_3 \vee \overline{M}_{k-2}$ and $H(k) = C_4 \vee \overline{C}_{k-2}$, respectively.

The inequality $\sqrt{x^2 + a} \leqslant x + \frac{a}{2x}$ yields the upper bounds. They approximate $\lambda_1(H(k))$ closely enough to be useful in practice. In particular, they simplify some of the inequalities in our arguments below. □

Because the expressions for $\lambda_1(H(k))$ agree with those for $\rho(k)$ in Theorem 2, it now remains to show that $\lambda_1(H(k)) \leqslant \lambda_1(H)$ for all graphs $H \in \mathcal{H}(k)$. To do this, we first prove (in the following lemma) that we may restrict our attention to graphs $H \in \mathcal{H}(k)$ of a specific order and size.

Lemma 7. *Let H be a graph in $\mathcal{H}(k)$ with $\lambda_1(H) = \rho(k)$. Then H has order n and size e where $n = k + 1$ if k is even, $n = k + 2$ if k is odd, and $2e = kn - k + 2$.*

Proof. Suppose that $2e > kn - k + 2$. Then, since n is odd, $2e \geqslant kn - k + 4$. Because the spectral radius of a graph is at least the average degree, $\lambda_1(H) \geqslant \frac{2e}{n} \geqslant k - \frac{k-4}{n}$. Noting that the final upper bound in Lemma 6 for odd $k \geqslant 5$ is at least as great as that for $k \geqslant 4$ and greater than $\theta = \lambda_1(H(3))$, we have

$$\lambda_1(H) - \rho(k) \geqslant \lambda_1(H) - \lambda_1\big(H(k)\big) > k - \frac{k-4}{k+1} - (k-1) - \frac{4}{k+1} > 0$$

and so $\lambda_1(H) \neq \rho(k)$. Thus, $2e = kn - k + 2$.

Because H has odd order n with maximum degree k, we have $n \geqslant k + 1$ if k is even and $n \geqslant k + 2$ if k is odd. If k is even and $n > k + 1$, then $k \geqslant 4$, $n \geqslant k + 3$, and it is straightforward to check that

$$\lambda_1(H) > \frac{2e}{n} = k - \frac{k-2}{n} \geqslant k - \frac{k-2}{k+3} > \frac{k - 2 + \sqrt{k^2 + 12}}{2} = \lambda_1\big(H(k)\big).$$

Thus $\lambda_1(H) > \rho(k)$, a contradiction. Hence, $n = k + 1$ if k is even. A similar argument shows that $n = k + 2$ if k is odd. □

Proof of Theorem 2. Let $\hat{\mathcal{H}}(k)$ denote the set of all graphs in $\mathcal{H}(k)$ that satisfy the order and size conditions in Lemma 7. By Lemma 7, to prove Theorem 2, it is sufficient to prove that $\lambda_1(H(k)) \leqslant \lambda_1(H)$ for each graph $H \in \hat{\mathcal{H}}(k)$. For then $\rho(k) = \lambda_1(H(k))$ and so $\rho(k)$ is given by the formula in Lemma 6. For even k the proof is straightforward. For odd k, we resort to a case analysis on the graph structure.

Throughout the argument, let H denote a graph in $\hat{\mathcal{H}}(k)$. It is straightforward to check that H must be a graph of maximum degree k obtained by deleting $(k-2)/2$ edges from the complete graph K_{k+1} when $k \geqslant 4$ is even and by deleting k edges from K_{k+2} when $k \geqslant 3$ is odd.

If k is even and $k \geqslant 4$, then by Lemma 7, H has order $n = k + 1$ and so has at least 3 vertices of degree $k = n - 1$. Let G_1 be the subgraph of H induced by $n_1 = 3$ of the vertices of degree k and let G_2 be the subgraph induced by the remaining $n_2 = n - 3$ vertices. Because each vertex in G_1 is adjacent to all other vertices in H, it follows that H has the same parameters $n_1, n_2, e_1, e_2, e_{12}$ in (4) as $H(k)$. Thus, for even $k \geqslant 4$, $\lambda_1(H(k)) \leqslant \lambda_1(H)$.

Suppose now that k is odd. By Lemma 7, H has order $n = k + 2$ and so has at least 4 vertices of degree $k = n - 2$. For $k = 3$, $H(3)$ is the only graph in $\hat{\mathcal{H}}(3)$, so we may assume that $k \geqslant 5$. Let G_1 be the subgraph of H induced by $n_1 = 4$ of the vertices of degree k and let G_2 be the subgraph induced by the remaining $n_2 = n - 4 = k - 2$ vertices.

Case 1. If G_1 may be chosen to be the complement of a perfect matching on 4 vertices, then $e_1 = 4$ and each vertex in G_1 is adjacent to each vertex in G_2. Thus H has the same parameters $n_1, n_2, e_1, e_2, e_{12}$ as $H(k)$ in (4), and so $\lambda_1(H) \geqslant \lambda_1(H(k))$ in this case.

Case 2. If G_1 may be chosen to be complete, then $e_1 = 6$ and each vertex in G_1 is adjacent to all but one vertex in G_2, so $e_{12} = n_1 n_2 - 4 = 4(k-3)$. Also, because $2e = kn - k + 2 = k^2 + k + 2$, it follows that $e_2 = e - e_1 - e_{12} = \frac{1}{2}(k^2 - 7k + 14)$. Substitution in (4) gives

$$\begin{aligned}\lambda_1(H) &\geqslant \frac{1}{2}\left(3 + \left(k - 5 + \frac{4}{k-2}\right)\right) + \frac{1}{2}\sqrt{\left(k - 8 + \frac{4}{k-2}\right)^2 + \frac{16(k-3)^2}{k-2}} \\ &= \frac{1}{2}\left(k - 2 + \frac{4}{k-2}\right) + \frac{1}{2}\sqrt{k^2 + 8 - \frac{32}{k-2} + \frac{16}{(k-2)^2}} \\ &> \frac{1}{2}\left(k - 3 + \sqrt{(k+1)^2 + 16}\right)\end{aligned}$$

where the last inequality follows by a straightforward calculation. Thus, in this case, $\lambda_1(H) \geqslant \lambda_1(H(k))$.

Suppose now that G_1 cannot be chosen to be as in Case 1 or Case 2 when k is odd, $k \geqslant 5$. Note that $H \in \hat{\mathcal{H}}(k)$ if and only if the complement $\overline{H}$ has order $n = k + 2$, size $\bar{e} = k = n - 2$, and no isolated vertices. Let $\overline{\varpi}$ be the number of components of $\overline{H}$. Each component has at least 2 vertices, since $\overline{H}$ has no isolated vertices. Because a spanning forest of $\overline{H}$ accounts for $n - \overline{\varpi}$ edges, it follows that $\overline{\varpi} \geqslant n - \bar{e} = 2$ and the remaining $\overline{\varpi} - 2$ edges are distributed among the $\overline{\varpi}$ components of $\overline{H}$. Thus, at least two of the components of $\overline{H}$ are trees. If three or more components are trees, we have Case 1 or Case 2. Thus, precisely two components of $\overline{H}$ are trees. If either one of the trees has 3 or more vertices of degree 1, we have Case 2. Thus, both trees have 2 vertices of degree 2 and so are paths. The remaining components (if any) must be cycles, or again we have Case 2. If both path components have order greater than 2, then again we have Case 2, while if both have order 2, then we have Case 1. It follows that if G_1 cannot be chosen to be in either Case 1 or Case 2 for odd $k \geqslant 5$, then we have the following case.

Case 3. The graph H is the complement of a disjoint union of K_2, a path P_m on $m \geqslant 3$ vertices and, if $n \geqslant m + 5$, a union C of cycles.

Assume first that $m \geqslant 5$. Consider the graph G on $k + 2$ vertices whose complement is the disjoint union of K_2, K_2, C_{m-2} and C, where C_{m-2} denotes a cycle on $m - 2 \geqslant 3$ vertices. By Case 1, we know that $\lambda_1(G) \geqslant \lambda_1(H(k))$.

Let x be the positive eigenvector of norm 1 corresponding to $\lambda_1(G)$. By using an equitable partition of G [13, p. 195], it follows that the entries of x are constant on the vertices of degree $k - 1$ in G (corresponding to the vertices on the cycles in $\overline{G}$) and constant and greatest on the 4 vertices of degree k (corresponding to the endpoints of the two K_2's in $\overline{G}$). Let 12 and 34 denote the two K_2's in $\overline{G}$. This means $12 \notin E(G)$ and $34 \notin E(G)$. Let 56 be an edge of the cycle C_{m-2} in $\overline{G}$. Similarly, this means $56 \notin E(G)$.

Note that the graph obtained from G by adding edges 34 and 56 to G and removing edges 35 and 46 from G is isomorphic to H. Also,

$$\begin{aligned}\lambda_1(H) \geqslant x^\top A(H)x = x^\top A(G)x + x^\top \big(A(H) - A(G)\big)x \\ &= \lambda_1(G) + 2(x_3x_4 + x_5x_6 - x_3x_5 - x_4x_6) \\ &= \lambda_1(G) + 2(x_3 - x_6)(x_4 - x_5) \\ &> \lambda_1(G).\end{aligned}$$

Since $\lambda_1(G) \geqslant \lambda_1(H(k))$, it follows that $\lambda_1(H) > \lambda_1(H(k))$ when $m \geqslant 5$.

Suppose now that $m = 3$. Partition the vertex set of $V(\overline{H})$ (and therefore of $V(H)$) into four parts: the two endpoints of K_2; the two endpoints of P_3; the internal vertex of P_3; and, the $k - 3$ vertices of C. This is an equitable partition of H with quotient matrix

$$\begin{bmatrix} 0 & 2 & 1 & k-3 \\ 2 & 1 & 0 & k-3 \\ 2 & 0 & 0 & k-3 \\ 2 & 2 & 1 & k-6 \end{bmatrix}.$$

Because the partition is equitable, a positive eigenvector of the quotient *lifts* [12, p. 198] to a positive eigenvector of H; that is, to a principal eigenvector. Thus the spectral radius of H equals the spectral radius of the quotient matrix. The characteristic polynomial of the quotient matrix is

$$\begin{aligned} P(x) &= x^4 - (k-5)x^3 - (4k-3)x^2 - (3k+7)x + 2k \\ &= \bigl(x^2 - (k-3)x - 2k - 2\bigr)\bigl(x^2 + 2x - 1\bigr) - 2. \end{aligned}$$

Since $\lambda_1 = \lambda_1(H)$ is a root of $P(x)$ and $\lambda_1 > 1$,

$$\lambda_1^2 - (k-3)\lambda_1 - 2k - 2 = \frac{2}{\lambda_1^2 + 2\lambda_1 - 1} > 0.$$

Because the polynomial $x^2 - (k-3)x - 2k - 2$ has roots $\lambda_1(H(k))$ and a negative number, it follows that $\lambda_1(H) = \lambda_1 > \lambda_1(H(k))$ when $m = 3$.

Suppose finally that $m = 4$. Partition the vertex set of $V(\overline{H})$ into four parts: the two endpoints of K_2; the two endpoints of P_4; the two internal vertices of P_4; and, the $k-4$ vertices of C. This is an equitable partition of H with quotient matrix

$$\begin{bmatrix} 0 & 2 & 2 & k-4 \\ 2 & 1 & 1 & k-4 \\ 2 & 1 & 0 & k-4 \\ 2 & 2 & 2 & k-7 \end{bmatrix}.$$

The characteristic polynomial of the quotient matrix is

$$\begin{aligned} Q(x) &= x^4 - (k-6)x^3 - (5k-8)x^2 - (7k+3)x - 2k - 4 \\ &= \bigl(x^2 - (k-3)x - 2k - 2\bigr)\bigl(x^2 + 3x + 1\bigr) - 2. \end{aligned}$$

Since $\lambda_1 = \lambda_1(H)$ is a root of $Q(x)$,

$$\lambda_1^2 - (k-3)\lambda_1 - 2k - 2 = \frac{2}{\lambda_1^2 + 3\lambda_1 + 1} > 0.$$

Because the polynomial $x^2 - (k-3)x - 2k - 2$ has roots $\lambda_1(H(k))$ and a negative number, we have $\lambda_1(H) = \lambda_1 > \lambda_1(H(k))$. This completes the proof that $\rho(k) = \lambda_1(H(k))$ and so establishes the formulas in Theorem 2. □

4. Factor-critical graphs

A graph G is factor-critical if for each $x \in V(G)$, the subgraph $G \setminus \{x\}$ has a perfect matching. This is a stronger property than $\nu(G) = \frac{n-1}{2}$. Gallai [11] (see also [21, Exercise 3.3.25, p. 147]) proved that G is factor-critical if and only if $|V(G)|$ is odd and

$$\operatorname{odd}(G \setminus S) \leqslant |S| \quad \text{for each nonempty subset } S \subset V(G). \tag{5}$$

We are now ready to prove Theorem 3.

Proof of Theorem 3. The proof is a refinement of that in Theorem 13 in [8]. Suppose that G satisfies the conditions of the theorem and, for some vertex x, $G \backslash \{x\}$ does not contain a perfect matching. Then G is not factor-critical, so by Gallai's condition (5), there is a nonempty subset $S \subset V(G)$ such that $q = \operatorname{odd}(G \setminus S) > |S| = s$. Thus, $q \geqslant s + 1$. Here, $s > 0$ since S is nonempty. Following the proof of

Lemma 5, we find that G contains two vertex disjoint subgraphs H_1, H_2 in $\mathcal{H}(k)$. Thus, as in (2), it follows from the inclusion principle that

$$\lambda_2(G) \geqslant \lambda_2(H_1 \cup H_2) \geqslant \min_i \lambda_1(H_i) \geqslant \rho(k),$$

a contradiction. $\square$

5. Graphs implying best bounds

The next lemma shows that the upper bound $\rho(k) = \lambda_1(H(k))$ in Theorem 1 is the best possible function of k when n is even. It also implies that $\rho(k)$ is still best possible when $\lambda_3(G)$ is replaced by $\lambda_2(G)$.

Lemma 8. *For each $k \geqslant 3$ there is a connected k-regular graph $G = G(k)$ of even order n such that*

$$\lambda_2(G) = \cdots = \lambda_k(G) = \rho(k), \tag{6}$$

yet $G(k)$ has no perfect matching.

Proof. Recall that $\rho(k) = \lambda_1(H(k))$ where $H(k)$ is defined in (3). Following Brouwer and Haemers [3], let $G(k)$ be the k-regular graph obtained by matching the $k-2$ vertices of degree $k-1$ in each of k copies of $H(k)$ to a set S of $|S| = k-2$ independent vertices. Then $G(k)\backslash S$ has $k > |S|$ copies of the odd order graph $H(k)$ as its components and so, by Tutte's theorem, $G(k)$ has no perfect matching.

We only prove the eigenvalue equalities (6) for the case where k is odd, $k \geqslant 5$. (The case of k even, $k \geqslant 4$, is similar and the case $k = 3$ is easily handled.)

Suppose then that k is odd, $k \geqslant 5$. If J denotes an all-ones matrix, then the vertices of $H = H(k)$ may be ordered so that it has partitioned adjacency matrix

$$A(H) = \begin{bmatrix} A(\bar{C}_{k-2}) & J \\ J^\top & A(C_4) \end{bmatrix}.$$

The $n = k^2 + 3k - 2$ vertices of G may then be ordered so that G has adjacency matrix:

$$A(G) = \begin{bmatrix} O & C & C & \cdots & C \\ C^\top & A(H) & O & \cdots & O \\ C^\top & O & A(H) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C^\top & O & O & \cdots & A(H) \end{bmatrix},$$

where each O denotes a zero matrix of the appropriate size and $C = [I \;\; O]$ where I is an identity matrix of order $k-2$.

The eigenvectors of $H(k)$ that are constant on each part of its two part equitable partition have eigenvalues given by the quotient matrix

$$\begin{bmatrix} k-5 & 4 \\ k-2 & 2 \end{bmatrix}.$$

These are $\frac{1}{2}(k - 3 \pm \sqrt{(k+1)^2 + 16})$ where, as observed in Lemma 6, the positive eigenvalue is $\lambda_1(H(k)) = \rho(k)$. Let x and y be eigenvectors of $H(k)$ associated with these two eigenvalues.

If u is a column eigenvector of $A(H)$, consider the $k-1$ column n-vectors

$$\left[0^\top, \ldots, 0^\top, u^\top, -u^\top, 0^\top, \ldots, 0^\top\right]^\top,$$

where the zero vectors are compatible with the partition of $A(G)$ and the first zero vector is always present. It is straightforward to check that these $k-1$ vectors are linearly independent eigenvectors of $A(G)$ with the same eigenvalue as u. Thus, each eigenvalue of $A(H)$ of multiplicity t yields an eigenvalue of $A(G)$ of multiplicity at least $t(k-1)$. In particular, taking $u=x$ and $u=y$, we see that the two eigenvalues of $A(H)$ above yield eigenvalues of $A(G)$ of multiplicity at least $k-1$ each. Thus, $\rho(k)=\lambda_1(H)$ is an eigenvalue of G with multiplicity at least $k-1$.

Now consider the $(2k+1)$-part equitable partition of $G(k)$ obtained by extending the 2-part partitions of the k copies of $H(k)$ in $G(k)$. Let W be the space consisting of n-vectors that are constant on each part of the partition. Then $\dim W = 2k+1$. Note that each of the $2(k-1)$ independent eigenvectors of $G(k)$ inherited from the eigenvectors x and y of $H(k)$ are in W. The natural 3-part equitable partition of $G(k)$ has quotient matrix

$$\begin{bmatrix} 0 & k & 0 \\ 1 & k-5 & 4 \\ 0 & k-2 & 2 \end{bmatrix}$$

with eigenvalues k and $(-3\pm\sqrt{17})/2$. Their corresponding eigenvectors lift to eigenvectors of $G(k)$ in W with the same eigenvalues, and these 3 eigenvalues are different from those above. Thus the three lifted eigenvectors, together with the previous $2(k-1)$ eigenvectors of $G(k)$ inherited from $H(k)$, form a basis for W.

The remaining eigenvectors in a basis of eigenvectors for $G(k)$ may be chosen orthogonal to the vectors in W; equivalently, they may be chosen to be orthogonal to the characteristic vectors of the parts of the $(2k+1)$-part partition because the characteristic vectors are also a basis for W. Consequently, they will be (some of the) eigenvectors of the matrix $A(\hat{G})$ obtained from $A(G)$ by replacing each all-ones block in each diagonal block $A(H)$ by an all-zeros matrix. But $A(\hat{G})$ is the adjacency matrix of a graph $\hat{G}$ with $k+1$ connected components, one of which is the graph G' obtained by attaching k copies of $\overline{C}_{k-2}$ to a set S of $k-2$ independent vertices by perfect matchings. Each of the remaining k components is a copy of C_4. It follows that the greatest eigenvalue of $\hat{G}$ is that of the component G'. Because G' has a two part equitable partition with quotient matrix

$$\begin{bmatrix} 0 & k \\ 1 & k-5 \end{bmatrix},$$

its greatest eigenvalue is $\frac{1}{2}(k-5+\sqrt{k^2-6k+25})$, and this is easily seen to be less than $k-1<\rho(k)$. □

The following lemma shows that the upper bound in Theorem 3 is the best possible function of k. Note that although the graph constructed is not factor-critical, it does have a matching covering $n-1$ vertices.

Lemma 9. *There is a connected k-regular graph $G'=G'(k)$ of odd order n for each (necessarily) even $k\geqslant 4$ such that*

$$\lambda_2(G')=\cdots=\lambda_{k-1}(G')=\rho(k),$$

yet $G'(k)$ is not factor-critical.

Proof. The construction is a variation of the construction of the graph $G(k)$ of even order in Lemma 8.

Let $G'(k)$ be the k-regular graph obtained by matching the $k-2$ vertices of degree $k-1$ in each of $k-1$ disjoint copies of $H(k)$ to the vertex set S of M_{k-2}. Then $G'(k)$ has $n=(k-1)(k+1)+k-2=k^2+k-3$ vertices, an odd number. Also, $\operatorname{odd}(G\setminus S)=k-1>|S|$, so by Gallai's condition (5), G is not factor-critical. However, using the techniques of Lemma 8, it can be shown that the eigenvalues satisfy the stated condition. □

6. Examples and comments

If G is a k-regular Ramanujan graph of even order n with $k \geqslant 6$, then $\lambda_2(G) \leqslant 2\sqrt{k-1} < k-1$, so G has at least $\lfloor (k - 2\sqrt{k-1} + 1)/2 \rfloor$ edge-disjoint perfect matchings by Corollary 4. (See Brandt et al. [2, Corollary 5.3] for the existence in G of a perfect matching and of long cycles when $k \geqslant 35$.)

Every k-regular graph G with $k \geqslant 3$ and with diameter at most 3 must have $\nu(G) = \lfloor n/2 \rfloor$. For if $\nu(G) < \lfloor n/2 \rfloor$, then G must have diameter greater than 3 because (as noted in [3, Corollary 3.4]), $t_i < n_i$ for $i \in \{1, 2, 3\}$ in the proof of Lemma 5.

The upper bounds on the eigenvalues $\lambda(G)$ in Theorems 1 and 3 hold when $\lambda(G)$ is an integer, because then $\lambda(G) \leqslant k-1$ since G is connected and so $\lambda(G) \leqslant \rho(k)$. This includes, for example, many distance regular graphs such as the Hamming graphs, the Johnson graphs, and the odd graphs, in particular, the Petersen graph. But, in fact, it is shown in [3] that every distance regular graph of even order has a perfect matching. It has yet to be determined whether or not every distance regular graph satisfies the conditions of Theorem 1.

If G is a vertex transitive graph and λ is a simple eigenvalue, then a result of Petersdorf and Sachs [20] (see also [4]) shows that λ must be an integer. Thus, if G is a connected vertex transitive graph of degree $k \geqslant 3$ then, by the comments above, Theorems 1 and 3 hold if the eigenvalues $\lambda(G)$ there are simple. But this is also a limited case, because the Gallai–Edmonds structure theorem [18, p. 94] implies that every vertex transitive graph of order n has a perfect matching if n is even and is factor-critical if n is odd. In particular, an abelian Cayley graph of degree $k \geqslant 3$ and order n is vertex transitive, but Theorem 1 can rarely be applied because, for fixed k, the spectral gap $k - \lambda_3$ approaches 0 as n increases (see [7], for example).

So far, we have examined bounds on the eigenvalues λ_2, λ_3. For eigenvalues λ_r with $3 \leqslant r < n$, it turns out that if G is a connected k-regular graph of order n with $k \geqslant 3$ then

$$\lambda_r(G) < \rho(k) \quad \text{implies that} \quad \nu(G) > \frac{n-r+1}{2}. \tag{7}$$

For, it is not difficult to check that if $\nu(G) \leqslant \frac{n-r+1}{2}$ in the proof of Lemma 5, then G has r vertex disjoint induced subgraphs $H_i \in \mathcal{H}(k)$ (see also [8, Theorem 11]), and so $\lambda_r(G) \geqslant \rho(k)$ as in (2).

Thus (7) implies that bounds on lower eigenvalues guarantee the existence of smaller matchings. For example, if the graph G in (7) has p positive eigenvalues then $\lambda_{p+1} \leqslant 0 < \rho(k)$ and so $\nu(G) > \frac{n-p}{2}$ if $p \geqslant 2$.

We showed in Section 5 that the bound in (7) is always best possible when $r = 2$ or $r = 3$. When $r \geqslant 4$, we have only been able to show that the bound is best possible for 3-regular graphs. Finding examples that show the bound in (7) is best for each $r \geqslant 4$ and $k \geqslant 4$ is likely more complicated and difficult.

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